

Exponential Moments of First Passage Times and Related Quantities for Random Walks

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Abstract

For a zero-delayed random walk on the real line, let $\tau(x)$, $N(x)$ and $\rho(x)$ denote the first passage time into the interval (x, ∞) , the number of visits to the interval $(-\infty, x]$ and the last exit time from $(-\infty, x]$, respectively. In the present paper, we provide ultimate criteria for the finiteness of exponential moments of these quantities. Moreover, whenever these moments are finite, we derive their asymptotic behaviour, as $x \rightarrow \infty$.

Keywords: first-passage time, last exit time, number of visits, random walk, renewal theory

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1 Introduction and main results

Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. real-valued random variables and $X := X_1$. Further, let $(S_n)_{n \geq 0}$ be the zero-delayed random walk with increments $S_n - S_{n-1} = X_n$, $n \geq 1$. For $x \in \mathbb{R}$, define the *first passage time* into (x, ∞)

$$\tau(x) := \inf\{n \in \mathbb{N}_0 : S_n > x\},$$

the *number of visits* to the interval $(-\infty, x]$

$$N(x) := \#\{n \in \mathbb{N} : S_n \leq x\} = \sum_{n \geq 1} \mathbb{1}_{\{S_n \leq x\}},$$

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and the *last exit time* from $(-\infty, x]$

$$\rho(x) := \begin{cases} \sup\{n \in \mathbb{N} : S_n \leq x\}, & \text{if } \inf_{n \geq 1} S_n \leq x, \\ 0, & \text{if } \inf_{n \geq 1} S_n > x. \end{cases}$$

Note that, for $x \geq 0$,

$$\rho(x) = \sup\{n \in \mathbb{N}_0 : S_n \leq x\}.$$

For typographical ease, throughout the text we write τ for $\tau(0)$, N for $N(0)$ and ρ for $\rho(0)$.

Our aim is to find criteria for the finiteness of the exponential moments of $\tau(x)$, $N(x)$ and $\rho(x)$, and to determine the asymptotic behaviour of these moments, as $x \rightarrow \infty$.

Assuming that $0 < \mathbb{E}X < \infty$, Heyde [11, Theorem 1] proved that

$$\mathbb{E}e^{a\tau(x)} < \infty \text{ for some } a > 0 \quad \text{iff} \quad \mathbb{E}e^{bX^-} < \infty \text{ for some } b > 0.$$

See also [3, Theorem 2] and [6, Theorem 2] for relevant results.

When $\mathbb{P}\{X \geq 0\} = 1$ and $\mathbb{P}\{X = 0\} < 1$,

$$\tau(x) - 1 = N(x) = \rho(x), \quad x \geq 0. \quad (1)$$

Plainly, in this case, criteria for all the three random variables are the same (Proposition 1.1). An intriguing consequence of our results in case when $\mathbb{P}\{X < 0\}\mathbb{P}\{X > 0\} > 0$, in which

$$\tau(x) - 1 \leq N(x) \leq \rho(x), \quad x \geq 0, \quad (2)$$

is that provided the abscissas of convergence of the moment generating functions of $\tau(x)$, $N(x)$ and $\rho(x)$ are positive there exists a unique value $R > 0$ such that *typically*

$$\mathbb{E}e^{a\tau(x)} < \infty, \mathbb{E}e^{aN(x)} < \infty \text{ iff } a \leq R, \text{ yet } \mathbb{E}e^{a\rho(x)} < \infty \text{ iff } a < R.$$

In particular, typically

$$\mathbb{E}e^{R\tau(x)} < \infty, \mathbb{E}e^{RN(x)} < \infty, \text{ but } \mathbb{E}e^{R\rho(x)} = \infty.$$

Also we prove that whenever the exponential moments are finite they exhibit the following asymptotics:

$$\mathbb{E}e^{a\tau(x)} \sim C_1 e^{\gamma x}, \mathbb{E}e^{aN(x)} \sim C_2 e^{\gamma x}, \mathbb{E}e^{a\rho(x)} \sim C_3 e^{\gamma x}, \quad x \rightarrow \infty,$$

for explicitly given $\gamma > 0$ and distinct positive constants C_i , $i = 1, 2, 3$ (when the law of X is lattice with span $\lambda > 0$ the limit is taken over $x \in \lambda\mathbb{N}$). Our results should be compared (or contrasted) to the known facts concerning power moments (see [13, Theorem 2.1 and Section 4.2] and [13, Theorem 2.2], respectively): for $p > 0$

$$\mathbb{E}(\tau(x))^{p+1} < \infty \quad \Leftrightarrow \quad \mathbb{E}(N(x))^p < \infty \quad \Leftrightarrow \quad \mathbb{E}(\rho(x))^p < \infty;$$

$$\mathbb{E}(\tau(x))^p \asymp \mathbb{E}(N(x))^p \asymp \mathbb{E}(\rho(x))^p \asymp \left(\frac{x}{\mathbb{E} \min(X^+, x)} \right)^p, \quad x \rightarrow \infty$$

where $f(x) \asymp g(x)$ means that $0 < \liminf_{x \rightarrow \infty} \frac{f(x)}{g(x)} \leq \limsup_{x \rightarrow \infty} \frac{f(x)}{g(x)} < \infty$.

Proposition 1.1 is due to Beljaev and Maksimov [2, Theorem 1]. A shorter proof can be found in [12, Theorem 2.1].

Proposition 1.1. *Assume that $\mathbb{P}\{X \geq 0\} = 1$ and let $\beta := \mathbb{P}\{X = 0\} \in [0, 1)$. Then for $a > 0$ the following conditions are equivalent:*

$$\mathbb{E}e^{a\tau(x)} < \infty \text{ for some (hence every) } x \geq 0;$$

$$a < -\log \beta$$

where $-\log \beta := \infty$ if $\beta = 0$. The same equivalence also holds for $N(x)$ and $\rho(x)$.

The following theorem provides sharp criteria for the finiteness of exponential moments of $\tau(x)$ and $N(x)$ in the case when $\mathbb{P}\{X < 0\} > 0$.

Theorem 1.2. *Let $a > 0$ and $\mathbb{P}\{X < 0\} > 0$. Then the following conditions are equivalent:*

$$\sum_{n \geq 1} \frac{e^{an}}{n} \mathbb{P}\{S_n \leq x\} < \infty \text{ for some (hence every) } x \geq 0; \quad (3)$$

$$\mathbb{E}e^{a\tau(x)} < \infty \text{ for some (hence every) } x \geq 0; \quad (4)$$

$$\mathbb{E}e^{aN(x)} < \infty \text{ for some (hence every) } x \geq 0; \quad (5)$$

$$a \leq R := -\log \inf_{t \geq 0} \mathbb{E}e^{-tX}. \quad (6)$$

Our next theorem provides the corresponding result for the last exit time $\rho(x)$.

Theorem 1.3. *Let $a > 0$ and $\mathbb{P}\{X < 0\} > 0$. Then the following conditions are equivalent:*

$$\sum_{n \geq 0} e^{an} \mathbb{P}\{S_n \leq x\} < \infty \text{ for some (hence every) } x \geq 0; \quad (7)$$

$$\mathbb{E}e^{a\rho(x)} < \infty \text{ for some (hence every) } x \geq 0; \quad (8)$$

$$a < R = -\log \inf_{t \geq 0} \mathbb{E}e^{-tX} \quad \text{or} \quad a = R \text{ and } \mathbb{E}Xe^{-\gamma_0 X} > 0 \quad (9)$$

where γ_0 is the unique positive number such that $\mathbb{E}e^{-\gamma_0 X} = e^{-R}$.

Now we turn our attention to the asymptotic behaviour of $\mathbb{E}e^{a\tau(x)}$, $\mathbb{E}e^{aN(x)}$ and $\mathbb{E}e^{a\rho(x)}$ and start by recalling a known result which, given in other terms, can be found in [12, Theorem 2.2]. In view of equality (1) we only state it for $\mathbb{E}e^{a\tau(x)}$. The phrase ‘ X is λ -lattice’ used in formulations of Proposition 1.4 and Theorem 1.5 is a shorthand for ‘The law of X is lattice with span $\lambda > 0$ ’.

Proposition 1.4. *Let $a > 0$, $\mathbb{P}\{X \geq 0\} = 1$ and $\mathbb{P}\{X = 0\} < 1$. Assume that $\mathbb{E}e^{a\tau(x)} < \infty$ for some (hence every) $x \geq 0$. Then, as $x \rightarrow \infty$,*

$$\mathbb{E}e^{a\tau(x)} \sim e^{\gamma x} \times \begin{cases} \frac{1-e^{-a}}{\gamma \mathbb{E}Xe^{-\gamma X}}, & \text{if } X \text{ is non-lattice,} \\ \frac{\lambda(1-e^{-a})}{(1-e^{-\lambda\gamma})\mathbb{E}Xe^{-\gamma X}}, & \text{if } X \text{ is } \lambda\text{-lattice} \end{cases}$$

where γ is a unique positive number such that $\mathbb{E}e^{-\gamma X} = e^{-a}$, and in the λ -lattice case the limit is taken over $x \in \lambda\mathbb{N}$.

Let

$$\varphi : [0, \infty) \rightarrow (0, \infty], \quad \varphi(t) := \mathbb{E}e^{-tX}$$

be the Laplace transform of X . When $0 < a \leq R$ and $\mathbb{P}\{X < 0\} > 0$, there exists a minimal $\gamma > 0$ such that $\varphi(\gamma) = e^{-a}$. This γ can be used to define a new probability measure \mathbb{P}_γ by

$$\mathbb{E}_\gamma h(S_0, \dots, S_n) = e^{an} \mathbb{E}e^{-\gamma S_n} h(S_0, \dots, S_n), \quad n \in \mathbb{N}, \quad (10)$$

for each nonnegative Borel function h on \mathbb{R}^{n+1} , where \mathbb{E}_γ denotes expectation with respect to \mathbb{P}_γ . Since $\mathbb{E}_\gamma X = \mathbb{E}_\gamma S_1 = -e^a \varphi'(\gamma)$ (where φ' denotes the left derivative of φ) and since φ is decreasing and convex on $[0, \gamma]$, there are only two possibilities:

$$\text{Either } \mathbb{E}_\gamma X \in (0, \infty) \quad \text{or} \quad \mathbb{E}_\gamma X = 0. \quad (11)$$

When $a < R$, then the first alternative in (11) prevails. When $a = R$, then typically $\varphi'(\gamma) = 0$ since γ is then unique minimizer of φ on $[0, \infty)$. In particular, $\mathbb{E}_\gamma X = 0$. But even if $a = R$ it can occur that $\mathbb{E}_\gamma X > 0$ or, equivalently, $\varphi'(\gamma) < 0$. Of course, then γ is the right endpoint of the interval $\{t \geq 0 : \varphi(t) < \infty\}$. We provide an example of this situation in Section 3.

Now we are ready to formulate the last result of the paper.

Theorem 1.5. *Let $a > 0$ and $\mathbb{P}\{X < 0\} > 0$.*

- (a) *Assume that $\mathbb{E}e^{a\tau(x)} < \infty$ for some (hence every) $x \geq 0$. Then $\mathbb{E}_\gamma S_\tau$ is positive and finite, and, as $x \rightarrow \infty$,*

$$\mathbb{E}e^{a\tau(x)} \sim e^{\gamma x} \times \begin{cases} \frac{\mathbb{E}(e^{a\tau}-1)}{\gamma \mathbb{E}_\gamma S_\tau}, & \text{if } X \text{ is non-lattice,} \\ \frac{\lambda \mathbb{E}(e^{a\tau}-1)}{(1-e^{-\lambda\gamma}) \mathbb{E}_\gamma S_\tau}, & \text{if } X \text{ is } \lambda\text{-lattice.} \end{cases} \quad (12)$$

- (b) *Assume that $\mathbb{E}e^{aN(x)} < \infty$ for some (hence every) $x \geq 0$. Then $\mathbb{E}_\gamma S_\tau$ is positive and finite, and, as $x \rightarrow \infty$,*

$$\mathbb{E}e^{aN(x)} \sim e^{\gamma x} \times \begin{cases} \frac{e^{-a} \mathbb{E}_\gamma \int_0^{S_\tau} e^{\gamma y} \mathbb{E}[e^{aN(-y)}] dy}{\mathbb{E}_\gamma S_\tau}, & \text{if } X \text{ is non-lattice,} \\ \frac{\lambda e^{-a} \mathbb{E}_\gamma \sum_{k=1}^{S_\tau/\lambda} e^{\gamma \lambda k} \mathbb{E}[e^{aN(-\lambda k)}]}{\mathbb{E}_\gamma S_\tau}, & \text{if } X \text{ is } \lambda\text{-lattice.} \end{cases} \quad (13)$$

- (c) *Assume that $\mathbb{E}e^{a\rho(x)} < \infty$ for some (hence every) $x \geq 0$. Then $M := \inf_{n \geq 1} S_n$ is positive with positive probability, and, as $x \rightarrow \infty$,*

$$\mathbb{E}e^{a\rho(x)} \sim e^{\gamma x} \times \begin{cases} \frac{e^{-a}(1-\mathbb{E}e^{-\gamma M^+})}{\gamma \mathbb{E}X e^{-\gamma X}}, & \text{if } X \text{ is non-lattice,} \\ \frac{\lambda e^{-a}(1-\mathbb{E}e^{-\gamma M^+})}{(1-e^{-\lambda\gamma}) \mathbb{E}X e^{-\gamma X}}, & \text{if } X \text{ is } \lambda\text{-lattice.} \end{cases} \quad (14)$$

In the λ -lattice case the limit is taken over $x \in \lambda\mathbb{N}$.

The rest of the paper is organized as follows. Section 2 is devoted to the proofs of Theorems 1.2, 1.3 and 1.5. In Section 3 we provide three examples illustrating our main results.

2 Proofs of the main results

Proof of Theorem 1.2. (6) \Rightarrow (3). Pick any $a \in (0, R]$ and let γ be as defined on p. 4. With this γ , the equality

$$Z_\gamma(A) := \sum_{n \geq 1} \frac{\mathbb{P}_\gamma\{S_n \in A\}}{n}$$

where $A \subset \mathbb{R}$ is a Borel set, defines a measure which is finite on bounded intervals. Furthermore, according to [1, Proposition 1.1 and Theorem 1.2], if $\mathbb{E}_\gamma X > 0$ then $Z_\gamma((-\infty, 0]) < \infty$, whereas if $\mathbb{E}_\gamma X = 0$ (this may only happen if $a = R$), then the function $x \mapsto Z_\gamma((-x, 0])$, $x > 0$, is of sublinear growth. Hence, for every $x \geq 0$,

$$\sum_{n \geq 1} \frac{e^{an}}{n} \mathbb{P}\{S_n \leq x\} = \sum_{n \geq 1} \frac{1}{n} \mathbb{E}_\gamma e^{\gamma S_n} \mathbb{1}_{\{S_n \leq x\}} = \int_{(-\infty, x]} e^{\gamma y} Z_\gamma(dy) < \infty.$$

(3) \Rightarrow (6). Suppose (3) holds for some $x = x_0 \geq 0$ and $a > R$. Pick $\varepsilon \in (0, a - R)$. Then $\sum_{n \geq 0} e^{(a-\varepsilon)n} \mathbb{P}\{S_n \leq x_0\} < \infty$ which is a contradiction to [12, Theorem 2.1(aiii)] (reproduced here as equivalence (7) \Leftrightarrow (9) of Theorem 1.3).

(3) \Rightarrow (4). The argument given below will also be used in the proof of Theorem 1.5.

If (3) holds for some $x \geq 0$ then, according to the already proved equivalence (3) \Leftrightarrow (6), first, $a \leq R$ and, secondly, (3) holds for every $x \geq 0$. For $0 < a \leq R$ and $x \geq 0$, we have

$$\begin{aligned} \mathbb{E} e^{a\tau(x)} &= 1 + (e^a - 1) \sum_{n \geq 0} e^{an} \mathbb{P}\{\tau(x) > n\} \\ &= 1 + (e^a - 1) \sum_{n \geq 0} e^{an} \mathbb{P}\{M_n \leq x\} \end{aligned} \quad (15)$$

where $M_n := \max_{0 \leq k \leq n} S_k$, $n \in \mathbb{N}_0$. According to [6, Formula (2.9)],

$$\sum_{n \geq 0} e^{an} \mathbb{P}\{M_n \leq x\} = \frac{\mathbb{E} e^{a\tau} - 1}{e^a - 1} \sum_{j \geq 0} e^{aj} \mathbb{P}\{L_j = j, S_j \leq x\} \quad (16)$$

where $L_j = \inf\{i \in \mathbb{N}_0 : S_i = M_j\}$, $j \in \mathbb{N}_0$. Since $a \leq R$, we can use the exponential measure transformation introduced in (10), which gives

$$e^{aj} \mathbb{P}\{L_j = j, S_j \leq x\} = \mathbb{E}_\gamma e^{\gamma S_j} \mathbb{1}_{\{L_j=j, S_j \leq x\}}.$$

Observe that $L_j = j$ holds iff $j = \sigma_k$ for some $k \in \mathbb{N}_0$ where σ_k ($\sigma_0 := 0$) denotes the k th strictly ascending ladder epoch of the random walk $(S_n)_{n \geq 0}$. Thus,

$$\begin{aligned} \sum_{j \geq 0} e^{aj} \mathbb{P}\{L_j = j, S_j \leq x\} &= \sum_{j \geq 0} \mathbb{E}_\gamma e^{\gamma S_j} \mathbb{1}_{\{L_j=j, S_j \leq x\}} \\ &= \sum_{j \geq 0} \mathbb{E}_\gamma \sum_{k \geq 0} e^{\gamma S_{\sigma_k}} \mathbb{1}_{\{\sigma_k=j, S_{\sigma_k} \leq x\}} = \mathbb{E}_\gamma \sum_{k \geq 0} e^{\gamma S_{\sigma_k}} \mathbb{1}_{\{S_{\sigma_k} \leq x\}} \\ &= e^{\gamma x} \int_{\mathbb{R}} e^{-\gamma(x-y)} \mathbb{1}_{[0, \infty)}(x-y) U_\gamma^>(dy) =: e^{\gamma x} Z_\gamma^>(x) \end{aligned} \quad (17)$$

where $U_\gamma^>$ denotes the renewal function of the random walk $(S_{\sigma_k})_{k \geq 0}$ under \mathbb{P}_γ , that is, $U_\gamma^>(\cdot) = \sum_{k \geq 0} \mathbb{P}_\gamma\{S_{\sigma_k} \in \cdot\}$. Thus, $Z_\gamma^>(x)$ is finite for all $x \geq 0$ since it is the integral of a directly Riemann integrable function with respect to $U_\gamma^>$.

(4) \Rightarrow (3) and (5) \Rightarrow (3). Since $\tau(y) \leq N(y) + 1$, $y \geq 0$, it suffices to prove the first implication. To this end, let

$$K(a) := \sum_{n \geq 1} \frac{e^{an}}{n} \mathbb{P}\{S_n \leq 0\}.$$

By a generalization of Spitzer's formula [6, Formula (2.6)], the assumption $\mathbb{E}e^{a\tau} < \infty$ immediately entails the finiteness of $K(a)$:

$$\infty > \mathbb{E}e^{a\tau} = 1 + (e^a - 1) \sum_{n \geq 0} e^{an} \mathbb{P}\{M_n = 0\} = 1 + (e^a - 1)e^{K(a)}.$$

We already know that if the series in (3) converges for $x = 0$, *i.e.*, if $K(a) < \infty$, then it converges for every $x \geq 0$.

(4) \Rightarrow (5). By the equivalence (3) \Leftrightarrow (4), $\mathbb{E}e^{a\tau(x)} < \infty$ for every $x \geq 0$. According to [13, Formula (3.54)],

$$\mathbb{P}\{N = k\} = \mathbb{P}\{\inf_{n \geq 1} S_n > 0\} \mathbb{P}\{\tau > k\}, \quad k \in \mathbb{N}_0,$$

where $\mathbb{P}\{\inf_{n \geq 1} S_n > 0\} > 0$, since, under the present assumptions, $(S_n)_{n \geq 0}$ drifts to $+\infty$ a.s. Hence, $\mathbb{E}e^{aN} < \infty$. Further, for $y \in \mathbb{R}$,

$$\widehat{N}(x, y) := \sum_{n > \tau(x)} \mathbb{1}_{\{S_n - S_{\tau(x)} \leq y\}} \quad (18)$$

is a copy of $N(y)$ that is independent of $(\tau(x), S_{\tau(x)})$. We have

$$N(x) = \tau(x) - 1 + \widehat{N}(x, x - S_{\tau(x)}) \leq \tau(x) + \widehat{N}(x, 0) \quad (19)$$

Hence, $\mathbb{E}e^{aN(x)} < \infty$, for every $x \geq 0$. The proof is complete. \square

Proof of Theorem 1.3. The equivalence (7) \Leftrightarrow (9) has been proved in [12, Theorem 2.1].

(7) \Rightarrow (8). According to the just mentioned equivalence, if (7) holds for some $x \geq 0$ it holds for every $x \geq 0$. It remains to note that for $x \geq 0$

$$\mathbb{P}\{\rho(x) = n\} = \int_{(-\infty, x]} \mathbb{P}\{\inf_{k \geq 1} S_k > x - y\} \mathbb{P}\{S_n \in dy\} \leq \mathbb{P}\{S_n \leq x\}. \quad (20)$$

(8) \Rightarrow (9). Suppose $\mathbb{E}e^{a\rho(x_0)} < \infty$ for some $x_0 \geq 0$ and $a > 0$. Since $\mathbb{E}e^{a\rho(x)}$ is increasing in x , we have $\mathbb{E}e^{a\rho} < \infty$. Condition $a \leq R$ must hold in view of (2) and implication (4) \Rightarrow (6) of Theorem 1.2. If $a < R$, we are done. In the case $a = R$ it remains to show that

$$\mathbb{E}Xe^{-\gamma_0 X} > 0. \quad (21)$$

Define the measure V by

$$V(A) := \sum_{n \geq 0} e^{Rn} \mathbb{P}\{S_n \in A\}, \quad (22)$$

for Borel sets $A \subset \mathbb{R}$. Then from (20) we infer that

$$\infty > \mathbb{E}e^{R\rho} = \int_{(-\infty, 0]} \mathbb{P}\{\inf_{n \geq 1} S_n > -y\} V(dy). \quad (23)$$

Under the present assumptions, the random walk $(S_n)_{n \geq 0}$ drifts to $+\infty$ a.s. Thus, $\mathbb{P}\{\inf_{n \geq 1} S_n > \varepsilon\} > 0$ for some $\varepsilon > 0$. With such an ε ,

$$\infty > \int_{(-\varepsilon, 0]} \mathbb{P}\{\inf_{n \geq 1} S_n > -y\} V(dy) \geq \mathbb{P}\{\inf_{n \geq 1} S_n > \varepsilon\} V((-\varepsilon, 0]).$$

Therefore,

$$\begin{aligned} \infty > V((-\varepsilon, 0]) &= \sum_{n=0}^{\infty} \mathbb{E}_{\gamma_0} e^{\gamma_0 S_n} \mathbb{1}_{\{-\varepsilon < S_n \leq 0\}} \\ &\geq e^{-\gamma_0 \varepsilon} \sum_{n=0}^{\infty} \mathbb{P}_{\gamma_0} \{-\varepsilon < S_n \leq 0\}. \end{aligned}$$

Hence $(S_n)_{n \geq 0}$ must be transient under \mathbb{P}_{γ_0} , which yields the validity of (9) in view of (11) and $\mathbb{E}_{\gamma_0} S_1 = e^R \mathbb{E}Xe^{-\gamma_0 X}$. The proof is complete. \square

Proof of Theorem 1.5. (a) In view of (15), (16) and (17), in order to find the asymptotics of $\mathbb{E}e^{a\tau(x)}$, it suffices to determine the asymptotic behaviour of $Z_\gamma^>(x)$ defined in (17). By the key renewal theorem on the positive half-line,

$$Z_\gamma^>(x) \xrightarrow{x \rightarrow \infty} \begin{cases} \frac{1}{\gamma \mathbb{E}_\gamma S_\tau} & \text{if } X \text{ is non-lattice,} \\ \frac{\lambda}{(1 - e^{-\lambda\gamma}) \mathbb{E}_\gamma S_\tau} & \text{if } X \text{ is } \lambda\text{-lattice} \end{cases} \quad (24)$$

where the limit $x \rightarrow \infty$ is taken over $x \in \lambda\mathbb{N}$ when X is lattice with span $\lambda > 0$.

It remains to check that $\mathbb{E}_\gamma S_\tau$ is finite. As pointed out in (11), either $\mathbb{E}_\gamma X \in (0, \infty)$ or $\mathbb{E}_\gamma X = 0$. In the first case, $S_n \rightarrow \infty$ a.s. under \mathbb{P}_γ and, therefore, $\mathbb{E}_\gamma \tau < \infty$, see, for instance, [4, Theorem 2, p. 151], which yields $\mathbb{E}_\gamma S_\tau < \infty$ by virtue of Wald's identity. If, on the other hand, $\mathbb{E}_\gamma X = 0$, then $\mathbb{E}_\gamma \tau = \infty$ and we cannot argue as above. But in this case, by [5, Formula (4a)], $\mathbb{E}_\gamma (S_1^+)^2 < \infty$ is sufficient for $\mathbb{E}_\gamma S_\tau < \infty$ to hold. Now the finiteness of

$$\mathbb{E}_\gamma e^{\gamma S_1} = \varphi(\gamma)^{-1} < \infty,$$

implies the finiteness of $\mathbb{E}_\gamma (S_1^+)^2$, and the proof of part (a) is complete.

(b) We only consider the case when X is non-lattice since the lattice case can be treated similarly. Denote by $R_x := S_{\tau(x)} - x$ the overshoot. Since $\mathbb{E} e^{a\tau(x)} = \mathbb{E}_\gamma e^{\gamma S_{\tau(x)}}$, we have in view of the already proved part (a)

$$\lim_{x \rightarrow \infty} \mathbb{E}_\gamma e^{\gamma R_x} = \frac{\mathbb{E} e^{a\tau} - 1}{\gamma \mathbb{E}_\gamma S_\tau}. \quad (25)$$

By Theorem 1.2, if $\mathbb{E} e^{aN(x)} < \infty$, then $\mathbb{E} e^{a\tau(x)} < \infty$. Therefore, according to part (a), we have $0 < \mathbb{E}_\gamma S_\tau < \infty$. This implies (see, for instance, [10, Theorem 10.3 on p. 103]) that, as $x \rightarrow \infty$, R_x converges in distribution to a random variable R_∞ satisfying

$$\mathbb{P}_\gamma \{R_\infty \leq x\} = \frac{1}{\mathbb{E}_\gamma S_\tau} \int_0^x \mathbb{P}_\gamma \{S_\tau > y\} dy, \quad x \geq 0.$$

In particular, under \mathbb{P}_γ , $e^{\gamma R_x}$ converges in distribution to $e^{\gamma R_\infty}$. Further,

$$\mathbb{E}_\gamma e^{\gamma R_\infty} = \frac{1}{\mathbb{E}_\gamma S_\tau} \int_0^\infty e^{\gamma y} \mathbb{P}_\gamma \{S_\tau > y\} dy = \frac{\mathbb{E}_\gamma e^{\gamma S_\tau} - 1}{\gamma \mathbb{E}_\gamma S_\tau} = \frac{\mathbb{E} e^{a\tau} - 1}{\gamma \mathbb{E}_\gamma S_\tau}.$$

Therefore, (25) can be rewritten as follows:

$$\lim_{x \rightarrow \infty} \mathbb{E}_\gamma e^{\gamma R_x} = \mathbb{E}_\gamma e^{\gamma R_\infty}. \quad (26)$$

Now we invoke a variant of Fatou's lemma sometimes called Pratt's lemma [14, Theorem 1]. To this end, note that, by a standard coupling argument, we can assume w.l.o.g. that $R_x \rightarrow R_\infty$ \mathbb{P}_γ -a.s. From (19) we infer that for $f(y) := \mathbb{E} e^{aN(y)}$, $y \in \mathbb{R}$ we have

$$f(x) = \mathbb{E} e^{aN(x)} = e^{-a} \mathbb{E} e^{a\tau(x)} f(-R_x) = e^{\gamma x} e^{-a} \mathbb{E}_\gamma e^{\gamma R_x} f(-R_x).$$

f is an increasing function and, therefore, has only countably many discontinuities. Hence $e^{\gamma R_x} f(-R_x)$ converges \mathbb{P}_γ -a.s. to $e^{\gamma R_\infty} f(-R_\infty)$. Further,

$$e^{\gamma R_x} f(-R_x) \leq e^{\gamma R_x} f(0)$$

and $e^{\gamma R_x} f(0)$ converges \mathbb{P}_γ -a.s. to $e^{\gamma R_\infty} f(0)$. Finally,

$$\lim_{x \rightarrow \infty} \mathbb{E}_\gamma e^{\gamma R_x} f(0) = \mathbb{E}_\gamma e^{\gamma R_\infty} f(0).$$

Therefore the assumptions of Pratt's lemma are fulfilled and an application of the lemma yields

$$\begin{aligned} \lim_{x \rightarrow \infty} e^{-\gamma x} f(x) &= e^{-a} \lim_{x \rightarrow \infty} \mathbb{E}_\gamma e^{\gamma R_x} f(-R_x) = e^{-a} \mathbb{E}_\gamma e^{\gamma R_\infty} f(-R_\infty) \\ &= \frac{e^{-a}}{\mathbb{E}_\gamma S_\tau} \int_0^\infty e^{\gamma y} f(-y) \mathbb{P}_\gamma\{S_\tau > y\} dy \\ &= \frac{e^{-a} \mathbb{E}_\gamma \int_0^{S_\tau} e^{\gamma y} f(-y) dy}{\mathbb{E}_\gamma S_\tau}. \end{aligned}$$

(c) From (20) and (22) (with R replaced by a and $M = \inf_{k \geq 1} S_k$), we infer

$$\begin{aligned} \mathbb{E} e^{a\rho(x)} &= \int_{(-\infty, x]} \mathbb{P}\{M > x - y\} V(dy) \\ &= V(x) \mathbb{P}\{M > 0\} - \int_{(0, \infty)} V(x - y) \mathbb{P}\{M \in dy\}, \quad x \geq 0. \end{aligned}$$

Assume that X is non-lattice and set $D_1 := \frac{e^{-a}}{\gamma \mathbb{E} X e^{-\gamma X}}$. It follows from (9) that $D_1 \in (0, \infty)$ and from [12, Theorem 2.2] that

$$V(x) \sim D_1 e^{\gamma x}, \quad x \rightarrow \infty. \tag{27}$$

The latter implies that for any $\varepsilon > 0$ there exists an $x_0 > 0$ such that

$$(D_1 - \varepsilon) e^{\gamma y} \leq V(y) \leq (D_1 + \varepsilon) e^{\gamma y}$$

for all $y \geq x_0$. Fix one such x_0 . Then for all $x \geq x_0$,

$$\begin{aligned} (D_1 - \varepsilon) e^{\gamma x} \int_{(0, x-x_0]} e^{-\gamma y} \mathbb{P}\{M \in dy\} &\leq \int_{(0, x-x_0]} V(x - y) \mathbb{P}\{M \in dy\} \\ &\leq (D_1 + \varepsilon) e^{\gamma x} \int_{(0, x-x_0]} e^{-\gamma y} \mathbb{P}\{M \in dy\}, \end{aligned}$$

and $\int_{(x-x_0, \infty)} V(x-y) \mathbb{P}\{M \in dy\} \in [0, V(x_0)]$. Letting first $x \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ we conclude that

$$\lim_{x \rightarrow \infty} e^{-\gamma x} \int_{(0, \infty)} V(x-y) \mathbb{P}\{M \in dy\} = D_1 \mathbb{E} e^{-\gamma M} \mathbb{1}_{\{M > 0\}}.$$

Together with (27) the latter yields

$$\begin{aligned} \mathbb{E} e^{a\rho(x)} &\sim D_1 (\mathbb{P}\{M > 0\} - \mathbb{E} e^{-\gamma M} \mathbb{1}_{\{M > 0\}}) e^{\gamma x} \\ &= D_1 (1 - \mathbb{E} e^{-\gamma M^+}) e^{\gamma x}, \quad x \rightarrow \infty. \end{aligned}$$

Under the present assumptions, the random walk $(S_n)_{n \geq 0}$ drifts to $+\infty$ a.s. Therefore, $\mathbb{P}\{M > 0\} > 0$ which implies that $1 - \mathbb{E} e^{-\gamma M^+} > 0$ and completes the proof in the non-lattice case.

The proof in the lattice case is based on the lattice version of [12, Theorem 2.2] and follows the same path. \square

3 Examples

In this section, retaining the notation of Section 1, we illustrate the results of Theorem 1.2 and Theorem 1.3 by three examples.

Example 3.1 (Simple random walk). Let $1/2 < p < 1$ and $\mathbb{P}\{X = 1\} = p = 1 - \mathbb{P}\{X = -1\} =: 1 - q$. Then the Laplace transform φ of X is given by $\varphi(t) = pe^{-t} + qe^t$ and $R = -\log(2\sqrt{pq})$. According to [8, Formula (3.7) on p. 272] and [7, Example 1], respectively,

$$\mathbb{P}\{\tau = 2n - 1\} = \frac{1}{2q} \frac{\binom{2n}{n}}{2^{2n} (2n - 1)} (2\sqrt{pq})^{2n}, \quad \mathbb{P}\{\tau = 2n\} = 0, \quad n \in \mathbb{N};$$

$$\mathbb{P}\{\rho = 2n\} = (p - q) \binom{2n}{n} (pq)^n, \quad \mathbb{P}\{\rho = 2n + 1\} = 0, \quad n \in \mathbb{N}_0.$$

Stirling's formula yields

$$\frac{\binom{2n}{n}}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}, \quad n \rightarrow \infty, \tag{28}$$

which implies that

$$\mathbb{E} e^{R\tau} < \infty \quad \text{and} \quad \mathbb{E} e^{R\rho} = \infty.$$

Example 3.2. Let $X \stackrel{d}{=} Y_1 - Y_2$ where Y_1 and Y_2 are independent r.v.'s with exponential distributions with parameters α and κ , respectively, $0 < \alpha < \kappa$. Then $\varphi(t) = \mathbb{E}e^{-tX} = \frac{\alpha\kappa}{(\alpha+t)(\kappa-t)}$ and $R = -\log(\frac{4\alpha\kappa}{(\alpha+\kappa)^2})$. According to [9, Formula (8.4) on p. 193], for $a \in (0, R)$,

$$\mathbb{E}e^{a\tau} = (2\alpha)^{-1}(\alpha + \kappa - \sqrt{(\alpha + \kappa)^2 - 4\alpha\kappa e^a}) < \infty.$$

Further, for $n \in \mathbb{N}_0$,

$$\begin{aligned} \mathbb{P}\{\rho = n\} &= \int_{(-\infty, 0]} \mathbb{P}\{\inf_{k \geq 1} S_k > -x\} \mathbb{P}\{S_n \in dx\} \\ &= \int_{(-\infty, 0]} \int_{(-x, \infty)} \mathbb{P}\{\inf_{k \geq 0} S_k > -x - y\} \mathbb{P}\{S_1 \in dy\} \mathbb{P}\{S_n \in dx\}. \end{aligned}$$

According to [9, Formula (5.9) on p. 410],

$$\mathbb{P}\{\inf_{k \geq 0} S_k > -x - y\} = \mathbb{P}\{\sup_{k \geq 0} (-S_k) < x + y\} = 1 - \frac{\alpha}{\kappa} e^{-(\kappa - \alpha)(x + y)}.$$

Note that S_n has the same law as the difference of two independent random variables with gamma distribution with parameters (n, α) and (n, κ) , respectively, which particularly implies that, for $x > 0$, the density of S_1 takes the form $\frac{\alpha\kappa e^{-\alpha x}}{\alpha + \kappa}$. Thus¹, for $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}\{\rho = n\} &= \int_{(-\infty, 0]} \int_{-x}^{\infty} \left(1 - \frac{\alpha}{\kappa} e^{-(\kappa - \alpha)(x + y)}\right) \frac{\alpha\kappa e^{-\alpha y}}{\alpha + \kappa} dy \mathbb{P}\{S_n \in dx\} \\ &= \frac{\kappa - \alpha}{\kappa} \int_{(-\infty, 0]} e^{\alpha x} \mathbb{P}\{S_n \in dx\} \\ &= \frac{\kappa - \alpha}{\kappa} \int_0^{\infty} \int_0^t e^{\alpha(s-t)} \frac{\alpha^n s^{n-1} e^{-\alpha s}}{(n-1)!} \frac{\kappa^n t^{n-1} e^{-\kappa t}}{(n-1)!} ds dt \\ &= \frac{\kappa - \alpha}{\kappa} \frac{\alpha^n \kappa^n}{n!(n-1)!} \int_0^{\infty} t^{2n-1} e^{-(\alpha + \kappa)t} dt \\ &= \frac{\kappa - \alpha}{(\kappa + \alpha)^{2n}} \alpha^n \kappa^{n-1} \binom{2n-1}{n}, \end{aligned}$$

and

$$\mathbb{P}\{\rho = 0\} = \frac{\kappa - \alpha}{\kappa}.$$

¹We do not claim that this formula is new, but we have not been able to locate it in the literature.

Hence,

$$\mathbb{E}e^{R\rho} = \frac{\kappa - \alpha}{\kappa} \left(1 + \sum_{n \geq 1} 4^{-n} \binom{2n-1}{n} \right) = \infty,$$

since relation (28) implies that the summands are of order $1/\sqrt{n}$, as $n \rightarrow \infty$.

Finally, we point out an explicit form of distribution of X for which $\mathbb{E}e^{R\rho(x)} < \infty$ for every $x \geq 0$.

Example 3.3. Fix $h > 0$ and take any probability law μ_1 on \mathbb{R} such that the Laplace-Stieltjes transform

$$\psi(t) := \int_{\mathbb{R}} e^{-tx} \mu_1(dx), \quad t \geq 0,$$

is finite for $0 \leq t \leq h$ and infinite for $t > h$, and the left derivative of ψ at h , $\psi'(h)$, is finite and positive. For instance, one can take

$$\mu_1(dx) := ce^{-h|x|}/(1+|x|^r)dx, \quad x \in \mathbb{R}$$

where $r > 2$ and $c := \left(\int_{\mathbb{R}} e^{-h|x|}(1+|x|^r)^{-1}dx \right)^{-1} > 0$.

Now choose s sufficiently large such that $\psi'(h) < s\psi(h)$. Then $\varphi(t) = e^{-st}\psi(t)$ is the Laplace-Stieltjes transform of the distribution $\mu := \delta_s * \mu_1$. Let X be a random variable with distribution μ . Plainly, $\varphi(t)$ is finite for $0 \leq t \leq h$ but infinite for $t > h$. Furthermore,

$$\varphi'(t) = e^{-st}(\psi'(t) - s\psi(t)), \quad |t| \leq h.$$

In particular, $\varphi'(h) < 0$ which, among other things, implies that $R = -\log \varphi(h)$ and that $\gamma_0 = h$. Therefore, $\mathbb{E}Xe^{-\gamma_0 X} = -\varphi'(h) > 0$, and by Theorem 1.2, $\mathbb{E}e^{R\rho(x)} < \infty$ for all $x \geq 0$.

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